Lévy flight of Photons in Hot Atomic Vapors

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CLXXIII International School of Physics "Enrico Fermi"
"Nano optics and atomics: transport of light and matter waves"
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Overview of this lecture:

1. Multiple Scattering of Light in Atomic Vapors
   1.1 Scattering Properties of Atoms
   1.2 Radiation Trapping of Light in Cold Atoms

2. Lévy Flight of Photons in Hot Atomic Vapors
   2.1 Random Walk
   2.2 Radiation Trapping of Light in Hot Atoms
   2.3 Step Size distribution of Photons
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1.1 Scattering Properties of Atoms

- Rayleigh scattering (elastic process) $\propto \omega^4$

- Mie scattering (elastic process)

- **Resonant scattering** (elastic and inelastic)
  - atoms at rest (😊)
  - moving atoms (Doppler)
  - multilevel atoms (Raman)
Photon Emission

Spontaneous Emission: \( \Gamma = 1/\tau_{\text{nat}} \)

\[ I_{\text{em}}(\omega) \propto \frac{1}{1 + 4\delta^2/\Gamma^2} \]

Optical Bloch Equation (RWA)

\[
\begin{align*}
\dot{\rho}_{ee} &= -\Gamma \tilde{\rho}_{ee} \\
\dot{\rho}_{eg} &= -\frac{\Gamma}{2} \tilde{\rho}_{eg} \\
\dot{\rho}_{gg} &= \Gamma \tilde{\rho}_{ee}
\end{align*}
\]

Initial condition \( \tilde{\rho}_{ee}(t = 0) = 1 \)

1.1 Scattering Properties of Atoms
Photon Emission

Optical Bloch Equation (RWA)

\[
\begin{align*}
\dot{\rho}_{ee} &= -\Gamma \rho_{ee} + i \frac{\Omega_L}{2} (\rho_{eg} - \rho_{ge}) \\
\dot{\rho}_{ge} &= -\left(i \delta_L + \frac{\Gamma}{2}\right) \rho_{ge} - i \frac{\Omega_L}{2} (\rho_{ee} - \rho_{gg}) \\
\dot{\rho}_{gg} &= \Gamma \rho_{ee} - i \frac{\Omega_L}{2} (\rho_{eg} - \rho_{ge}) \\
\end{align*}
\]

initial condition \( \rho_{ee}(t = 0) = 1 \)

Stimulated Emission

\[\hbar\Omega_L = -dE\]

\[s_0 = \frac{2\Omega_L^2}{\Gamma^2} = \frac{I}{I_{sat}}\]
Photon Absorption

Optical Bloch Equation

\[ \dot{\tilde{\rho}}_{ee} = -\Gamma \tilde{\rho}_{ee} + i \frac{\Omega_L}{2} (\tilde{\rho}_{eg} - \tilde{\rho}_{ge}) \]

\[ \dot{\tilde{\rho}}_{ge} = -(i\delta_L + \frac{\Gamma}{2}) \tilde{\rho}_{ge} - i \frac{\Omega_L}{2} (\tilde{\rho}_{ee} - \tilde{\rho}_{gg}) \]

\[ \dot{\tilde{\rho}}_{gg} = \Gamma \tilde{\rho}_{ee} - i \frac{\Omega_L}{2} (\tilde{\rho}_{eg} - \tilde{\rho}_{ge}) \]

initial condition \( \tilde{\rho}_{gg}(t = 0) = 1 \)

absorption spectrum = emission spectrum

1.1 Scattering Properties of Atoms
scattering of photon ≠ absorption + emission !!!

\[ |e> \rightarrow |g> \]

Induced dipole

\[ <d> = \text{Tr}(\rho d) = ||d||(\rho_{ge} - \rho_{eg}) \]

\[ d = ||d||(|e><g| + |g><e|) \]

index of refraction:

\[ n = 1 - \rho \frac{6\pi}{k^3} \frac{\delta/\Gamma}{1+4(\delta/\Gamma)^2} + i\rho \frac{6\pi}{k^3} \frac{1}{1+4(\delta/\Gamma)^2} \]

Scattered field

\[ E_{sc} \propto <d> \]

\[ d = \alpha E_L \]

\[ \alpha = Re(\alpha) + iIm(\alpha) \]

\[ n - 1 < 10^{-4} \text{ (MOT)} \]

\[ n - 1 < 0.1 \text{ (BEC)} \]

\[ \Rightarrow \text{ elastic scattering} \]

1.1 Scattering Properties of Atoms
Photon Scattering

Spectrum of Scattered Field

elastic scattering for
- cold atoms ($kv \ll \Gamma$)
- low incident intensity ($I \ll I_{\text{sat}}$)
- no internal structure (Raman scattering)
1.1 Scattering Properties of Atoms

Inelastic Scattering

- residual Doppler broadening
- power broadening and Mollow triplet

\[ I_{sc} = I_{coh} + I_{incoh} \propto \frac{1}{2} \frac{s}{1+s} \]

\[ I_{incoh} \propto \frac{1}{2} \frac{s^2}{(1+s)^2} \]

\[ I_{coh} \propto \frac{1}{2} \frac{s}{(1+s)^2} \]
Photon Scattering

1.1 Scattering Properties of Atoms

Inelastic Scattering

Hot Atoms: Doppler broadening

\[ k \nu \gg \Gamma \]

\[ k \nu < \Gamma \]
1.1 Scattering Properties of Atoms

Inelastic Scattering

Internal Structure: Raman transitions

- Raman Stokes
- Raman anti-Stokes
1.1 Scattering Properties of Atoms

Optical Resonances of glass

UV

visible

index of refraction

'anomalous dispersion'
1.1 Scattering Properties of Atoms

Energy levels

\[ \dot{\rho}_{ee} = - \frac{\tilde{\rho}_{ee}}{T_1} \]
\[ \dot{\rho}_{eg} = - \frac{\tilde{\rho}_{eg}}{T_2} \]
\[ \dot{\rho}_{gg} = \frac{\tilde{\rho}_{ee}}{T_1} \]

purely radiative decay: \( T_1 = \frac{T_2}{2} = \frac{1}{\Gamma} \)

collisions / phonons: \( T_1 \gg T_2 \)
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1.2 Radiation Trapping of Light in Cold Atoms

Single scattering (\(b<<1\))

\[ \ell_{\text{scat}} = \frac{1}{n\sigma} \]

Beer-Lambert law: \(T_{\text{coh}} = e^{-b} = e^{-n\sigma L}\)

Multiple scattering (\(b>>1\))

Ohm’s law: \(T_{\text{diff}} \propto \ell_{\text{scat}} / L\)
1.2 Radiation Trapping of Light in Cold Atoms

‘Static’ Experiments

probe beam

cold atoms

diffuse reflexion
diffuse transmission

MOT parameters:
MOT diameter ≈ few mm
N ≈ few 10^9
b ≈ 20
velocities ≈ 0.1 m/s
1.2 Radiation Trapping of Light in Cold Atoms

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase velocity</td>
<td>$c = \frac{c_0}{n}$</td>
<td>propagation of phase for a monochromatic wave $c &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$c \ll c_0$</td>
<td></td>
</tr>
<tr>
<td>Group velocity</td>
<td>$v_g = \frac{\partial \omega}{\partial k}$</td>
<td>propagation of transmitted gaussian pulse with slowly varying envelope $v_g &lt; 0$</td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>v_g</td>
</tr>
<tr>
<td>Transport velocity</td>
<td>$0 &lt; v_{tr} &lt; c_0$</td>
<td>propagation of scattered wave energy</td>
</tr>
</tbody>
</table>

![Diagram](image.png)
Diffusion Theory:

slab: \( T(t) \propto \Sigma_n (-1)^{n+1} n^2 e^{-n^2 \pi^2 D t/L^2} \)

late decay time \( \propto e^{-t/\tau_0} \)

\[ \tau_0 \approx \frac{L^2}{\pi^2 D} = \frac{3}{\pi^2} b^2 \tau_{tr} \]

\[ b = \frac{L}{l} \text{ optical thickness} \]

\[ D = \frac{1}{3} \frac{l_{tr}^2}{\tau_{tr}} \text{ transport mean-free path} \]

\[ = \frac{1}{3} l_{tr} v_{tr} \text{ transport time} \]

\[ = \frac{1}{3} l_{tr} \text{ transport velocity} \]
\[ \tau_0 \approx \frac{L^2}{\pi^2 D} \Rightarrow D \approx 0.66 \text{m}^2/\text{s} \]

\[ \frac{V_{\text{tr}}}{c_0} = \frac{l_{\text{tr}}}{c_0 \tau_{\text{tr}}} \approx 3 \cdot 10^{-5} \]

D: smaller than in Ti02 with kl \( \approx 1 \) (D\( \approx 4 \text{m}^2/\text{s} \))

NOT due to interferences in multiple scattering (\( \neq \) Localization)

1.2 Radiation Trapping of Light in Cold Atoms
\[ \tau_0 \approx \frac{L^2}{\pi^2 D} = \frac{3}{\pi^2} b^2 \tau_{\text{tr}} \]

**Transport Time:**

\[ \tau_{\text{tr}} \approx \tau_{\text{Wigner}}(\delta) + \frac{l(\delta)}{v_{\text{gr}}(\delta)} \]

**N = cte**

**b = cte = 11**
From Coherent to Incoherent Radiation Trapping

$k v \ll \ll \ll \ll \Gamma$

atoms at rest

Monte Carlo Simulation

Experiments

atomic motion → 3 regimes:

- elastic
  - $\tau_0 \propto b^2 \tau_{tr}$

- frequency diffusion
  - $\tau_0 = f(b \, kv/\Gamma)$

- Doppler
  - $\tau_0 \propto b \, [\log(b)]^{1/2}$

1.2 Radiation Trapping of Light in Cold Atoms
From Coherent to Incoherent Radiation Trapping

± cold atoms: Doppler shift of scattered photons
\[ \Delta \nu \sim \sqrt{D \nu t} \sim \sqrt{\nu^2 b^2 \tau_{\text{nat}}} \ll \Gamma \]

\[ \tau_{\nu=0} \propto \left( \frac{L}{l_0} \right)^2 \Gamma^{-1} \]

\[ \tau_{\text{Holstein}} \propto \left( \frac{L}{l_0} \right) \sqrt{\log \left[ \frac{L}{(2l_0)} \right]} \Gamma^{-1} \]

CFR: complete frequency redistribution

Monte Carlo Simulation

From R. Pierrat et al., arXiv:0904.0936v1

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Normal Diffusion: \[ \Delta x^2 = \frac{1}{2} D t \]

\[ D \propto \frac{l^2}{\tau} \]
For a step size distribution $P(x)$:

\[ l = \langle x \rangle = \frac{\int dx x P(x)}{\int dx P(x)} \quad D = \langle x^2 \rangle = \frac{\int dx x^2 P(x)}{\int dx P(x)} \]

Example:

\[ P(x) \propto \exp(-x/l_0) \Rightarrow l = \langle x \rangle = l_0 \]

\[ D \propto \langle x^2 \rangle \propto l_0^2 \]

if $\langle x \rangle$ and $\langle x^2 \rangle$ are finite
the distribution of a large number of steps $\Sigma x_i$ converges to a Gaussian distribution
(Central Limit Theorem)
Random Walk of Step Size Distribution

\[ P(x) \propto \frac{1}{x^\alpha} \]

<table>
<thead>
<tr>
<th>nonstationary</th>
<th>‘ballistic’</th>
<th>‘super-diffusive’</th>
<th>normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \int P(x)dx = \infty ]</td>
<td>[ \langle x \rangle = \infty ]</td>
<td>[ \langle x^2 \rangle = \infty ]</td>
<td>[ \langle x^2 \rangle &lt; \infty ]</td>
</tr>
</tbody>
</table>

\[ \text{Lévy Flights} \]

how to measure \[ \langle x \rangle \pm \sqrt{\langle x^2 \rangle} \]: weak ergodicity breaking

2.1 Random Walk
Lévy Flights in

2.1 Random Walk
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Multiple Scattering of Light in Hot Atomic vapors:

Random walk of photons / Radiation trapping in

- dense atomic vapours
- discharge
- hot plasmas
- gas lasers
- stars
- intergalactic scattering
Milne Equation

\[ \nabla^2 \left[ n(r, t) + \tau \frac{\partial n(r, t)}{\partial t} \right] = 4k^2 \tau \frac{\partial n(r, t)}{\partial t} \]

§ diffusion equation for excited state population (and photons)

nice idea:

estimation of photon escape time from the sun

\[ L_{\text{sun}} = 10^6 \text{ km} = 10^9 \text{ m} \quad \text{and} \quad l = 1 \text{ mm} = 0.001 \text{ m} \]

\[ D = \frac{lc}{3} \quad L_{\text{sun}}^2 = D \ t_{\text{escape}} \quad \Rightarrow \ t_{\text{escape}} = \frac{3 \times 10^{18}}{\left(10^{-3} \times 3 \times 10^8\right)} = 10^{13} \text{ s} = 300 \text{ 000 years} \]

nice but: wrong!!! (assumes box-shaped atomic lineshape: no wings)

(±used in many estimation of photon life time in the sun I have seen)

A random walk of photons in hot atomic vapours
is NOT correctly described by a diffusion equation

2.2 Radiation Trapping of Light in Hot Atoms
On Radiation Diffusion and the Rapidity of Escape of Resonance Radiation from a Gas

By Carl Kent

General Electric Vapor Lamp Company, Hoboken, N. J.

(Received August 25, 1932)

The radiation diffusion process is considered from the standpoint of the free paths of the diffusing resonance quanta as influenced by the Doppler and other line broadening effects. Abnormally long free paths are found to be of such importance as to enable resonance radiation to escape from a body of gas faster than has usually been supposed. It is assumed that a large concentration of diffusing resonance quanta will, on the basis of Doppler broadening only, give rise to a characteristic excitation of atoms, as dependent on their speeds, which can be represented by a distribution function which will lie between two limiting distribution functions, namely (1) Maxwell's distribution function and (2) a distribution function expressing a lower relative excitation of the high speed atoms than that of Maxwell, based on the excitation of all atoms as if by absorption of the core of the line. On the basis of (1) and (2), limiting expressions are derived for: (a) the fraction of emitted quanta traversing at least a given distance before absorption, (b) the diffusion coefficient, (c) the average square free path, (d) the average free path. A fundamental difference between radiation diffusion and molecular diffusion appears in that whereas (a) decreases exponentially with the distance in the latter case it is found to decrease only linearly (roughly) with the distance in the former case. For this reason very long free paths are found to be of relatively great importance in radiation diffusion. It is found that, for a gas container of infinite size, (b), (c), and (d) are all infinite. For a gas container of finite size, esti-
Holstein Equation

\[ \frac{\partial n(r, t)}{\partial t} = -\frac{1}{\tau} n(r, t) + \frac{1}{\tau} \int_V n(r', t) G(r, r') dr' \]

modal expansion of \( n(r,t) \)  
\[ \text{estimation of escape factors of modes} \]

extensively studied for time dependant photon escape in a large variety of situations

Important ingredient \( G(r,r') \):
\[ \text{i.e. how far flies a photon between two successive scattering events} \]

\[ P(x) = ? \]
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Photon Trajectories in Incoherent Atomic Radiation Trapping as Lévy Flights

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(Received 19 November 2003; published 13 September 2004)

Photon trajectories in incoherent radiation trapping for Doppler, Lorentz, and Voigt line shapes under complete frequency redistribution are shown to be Lévy flights. The jump length (r) distributions display characteristic long tails. For the Lorentz line shape, the asymptotic form is a strict power law $r^{-3/2}$, while for Doppler the asymptotic is $r^{-2}(\ln r)^{-1/2}$. For the Voigt profile, the asymptotic form always has a Lorentz character, but the trajectory is a self-affine fractal with two characteristic Hausdorff scaling exponents.

2.3 Step Size Distribution of Photons
\( T(x) = \frac{I(L)}{I(0)} \)

\[
T(x) = \langle e^{-x/l(\omega)} \rangle = \int d\omega f_{inc}(\omega)e^{-f_{abs}(\omega)x}
\]

\[P(x) = \frac{\partial T(x)}{\partial x} = \int d\omega f_{inc}(\omega)f_{abs}(\omega)e^{-f_{abs}(\omega)x}\]

everything depends on the precise forms of
\( l(\omega) / f_{abs}(\omega) \) and \( f_{inc}(\omega) / f_{em}(\omega) \)

2.3 Step Size Distribution of Photons
Example: pure Doppler emission and absorption lines assume $f_{em} = f_{abs}$ (Complete Frequency Distribution)

$$P(x) \propto \frac{1}{x^2 \sqrt{\ln(x)}}$$

$\langle x \rangle$ is finite

$$\langle x^2 \rangle \propto \int dx x^2 P(x) = \infty$$

superdiffusion / Lévy flight

2.3 Step Size Distribution of Photons
How to measure $P(x)$?

How to track a Photon?

In Stars … 😞

in the lab …

2.3 Step Size Distribution of Photons
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2.3 Step Size Distribution of Photons
$P(x) \propto \int d\omega f(\omega) e^{-n\sigma(\omega)x}$

$P(x, \omega_0) \propto e^{-n\sigma(\omega_0)x}$

**Power law**

**Beer-Lambert**

2.3 Step Size Distribution of Photons
neglecting the natural width of the atoms: $\alpha = 2$

2.3 Step Size Distribution of Photons
Spatial evolution of the spectrum

2.3 Step Size Distribution of Photons
Multilevel calculation of spectral evolution

2.3 Step Size Distribution of Photons
Steady state random walk:
P(x) after many scattering events
(to forget initial memory / partial frequency distribution)

signal/noise limit: photon shot noise, cosmic rays!
( combine 6 images of 5 hours each )

2.3 Step Size Distribution of Photons
$\alpha < 3 \therefore$ Experimental evidence of heavy tail $P(x) \therefore$ Lévy flight of photons

2.3 Step Size Distribution of Photons
Lévy Flights with atomic vapours:

Further developments:

- Tune the power law exponent?
  (detuning, magnetic field, saturation, collisional broadening)

- Truncated Lévy Flights
  (spatial, frequency)

- Ergodicity: \( x^2 \propto t^\gamma \)
  (time resolved exp.)
Conclusions

• Light scattering by atoms is more than
  – spontaneous emission
  – classical dipole emission

• Radiation trapping in atomic vapours
  – cold atoms : slow diffusion
  – hot atoms : Lévy flights

Be careful when using average values!